Secondary School Students Investigating Mathematics

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This paper describes a research study to find out the ability of Singapore secondary school students in attempting open investigative tasks. The results show that most high-ability students had no experience in open mathematical investigation and they did not even know how to start. Providing sample problems in the tasks for students to investigate did not seem to help them understand the requirements of the tasks. The implication of these findings on research methodology using paper-and-pencil tests will be discussed.

Many school mathematics curricula emphasise the use of mathematical investigation in the teaching and learning of mathematics. For example, the Australian national curriculum states that, "Mathematical investigations can help students to develop mathematical concepts and can also provide them with experience of some of the processes through which mathematical ideas are generated and tested" (Australian Education Council, 1991, p. 14) and the Cockcroft Report in the United Kingdom stipulates that "mathematics teaching at all levels should include opportunities for … investigational work" (Cockcroft, 1982, p. 71). In New Zealand, their Mathematics in the New Zealand Curriculum (Ministry of Education of New Zealand, 1992) also stresses the importance of mathematical investigation.

So what exactly is mathematical investigation and why do many school mathematics curricula place great emphasis on it? There is a long-standing debate about the differences between problem solving and investigation but Pirie (1987) alleged that no fruitful service would be performed by indulging in the 'investigation' versus 'problem solving' dispute. However, Frobisher (1994) believed that this was a crucial issue which would affect how and what educators teach their students. Evans (1987) claimed that Cockcroft (1982) made a distinction although the latter did not say what it was. Some educators (e.g., Orton & Frobisher, 1996) perpetuated the idea that an investigative task must be open, i.e., it should not specify any problem for the students to solve or investigate; students must pose their own problems and thus mathematical investigation involves both problem posing and problem solving. For example, Bastow, Hughes, Kissane and Mortlock (1991) defined mathematical investigation as the "systematic exploration of *open* [emphasis mine] situations that have mathematical features" (p. 1) while Ernest (1991) described problem solving as "trail-blazing to a desired location" (p. 285) and investigation as the exploration of an unknown land where "the journey, not the destination, is the goal" (Pirie, 1987, p. 2). So problem solving is a convergent activity with a well-defined goal and answer while investigation is a divergent activity with an open goal and answer (Evans, 1987).

The use of open investigative tasks helps students to focus on "the process of problem solving and the open-endedness of a problem or investigation" (Hawera, 2006, p. 286) because many genuine problems are open and ill-structured in nature. In real life, no one will tell you exactly what the problem is and what the boundaries of the problem are (Simon, 1973). You have to find the root problem first before you can resolve the issue. Similarly, in mathematical investigation, the task will not tell the students what the problems are. The students will have to think through and pose their own problems to solve or investigate. This helps the students to be more aware of the problem situation and to take charge of the issue at hand. Many educators (e.g., Brenner & Moschkovich, 2002) also favour bringing academic mathematicians' practices into the classroom and this includes letting students engage in a variety of rich mathematical activities that parallel what academic mathematicians do. And what do mathematicians do? They investigate and solve mathematical problems (Civil, 2002). Lampert (1990) believed that such activities encourage students to think mathematically, such as problem posing (Brown & Walter, 2005), conjecturing and generalising (Calder, Brown, Hanley, & Darby, 2006).

In this paper, I will describe a research study on open mathematical investigation performed by some high-ability secondary school students in Singapore. It will begin with the background of the study, followed by the research methodology and findings, and it will end with a discussion of the data collected and some implications for both teaching and research.

Background

The central theme of the Singapore mathematics curriculum is mathematical problem solving (Ministry of Education of Singapore, 1990) and most mathematics teachers in Singapore are familiar with solving mathematical problems. Although the curriculum specifies that "a problem covers a wide range of situations from routine mathematical problems to problems in unfamiliar context and open-ended investigations" (Ministry of Education of Singapore, 2000, p. 10), many teachers are not sure what mathematical investigation involves. Whenever I mention the term 'mathematical investigation', quite a number of teachers will look at me blankly and ask, "What's that?" Some teachers have this vague idea that mathematical investigation has something to do with guided-discovery learning, but, according to Ernest (1991), there are some major differences. Very few teachers actually know what open investigative tasks are, and when faced with such a task, most, if not all, of them do not know what to do (this was gathered from courses conducted by me for teachers). If most teachers are not familiar with open mathematical investigation, then it is unlikely that they will teach their students how to deal with open investigative tasks.

As explained in the previous section, learning how to investigate using open investigative tasks is very important to cultivate important mathematical processes. So there is a need to find out the current state of competency in open mathematical investigation among students in Singapore, and to develop a teaching programme to help students learn how to deal with open investigative tasks. This paper reports a research study to find out the current state of proficiency in open mathematical investigation among high-ability secondary school students and is part of a larger study that researches on the nature and development of thinking processes when students with a wide range of mathematical abilities attempt open investigative tasks.

Methodology

The sample consisted of 29 Secondary One students from an intact class. This class of high-ability students was selected randomly from one of the top schools in Singapore. From the written survey conducted at the end of the written test, all of these students said that they had not seen this kind of open investigative tasks before. The paper-and-pencil test consisted of four open investigative tasks to be completed individually within one hour and thirty minutes. Because of anecdotal evidence that many teachers and students might not know how to begin when faced with such open tasks (see previous section), the first task included sample problems for students to investigate if they did not know how to pose their own problems (see below). Subsequent tasks were completely open with no hints or guidance. The topic for the tasks was arithmetic because Secondary One students were most familiar with arithmetic: If another topic, such as algebra, was chosen, then if the students did not know what to do, it might be because they were unfamiliar with the topic rather than with the investigation itself. The test was administered by the author himself.

Mathematical Investigative Task 1: Powers of a Number

9^5 means 9 multiplied by itself 5 times, i.e., $9^5 = 9 \times 9 \times 9 \times 9 \times 9 = 59\,049$.

Powers of 9 are 9^1, 9^2, 9^3, 9^4, 9^5, 9^6, … etc.

Investigate the powers of 9.

For example, you can investigate the following or you can *pose your own problems to investigate*:

a) Find as many patterns as you can about the powers of 9.

b) Explain why these patterns occur.

c) Do these patterns occur for powers of other numbers?

Findings

Within five minutes from the start of the written test, five students raised their hand and asked me what they were supposed to do for the first investigative task. Some students did not know what to do but they did not ask me. The following are what some of these students (all names are pseudonyms) wrote in their answer scripts or in the written survey.

Albert: I am thinking of asking the teacher what ‘investigate’ means.

Ben: I find it a bit difficult as I do not understand the meaning of investigate.

In order to analyse how and what the students investigated, two rubrics were developed. Table 1 shows the first rubric which describes *how* students investigated the first task. There are five levels. In Level 1H0 (the first number refers to the task number and the letter H stands for How), the students did not do anything. In Level 1H1, the students wrote something superficial or irrelevant that did not contribute anything to the investigation. I was surprised that some students just concluded, “The pattern is 9^1, 9^2, 9^3, 9^4, 9^5 which will continue.” These students did not find the numerical values of the powers, thus suggesting that they did not know how to investigate by examining specific cases. In Level 1H2, the students did list out the numerical values but they did not do anything after that. In Level 1H3, the students tried to find some patterns but there was nothing constructive. In Level 1H4, the students did find some patterns.

The analysis shows that about 35% of the students did not know how to investigate by examining empirical data (Levels 1H0 and 1H1). About 17% of the students did try something but were unable to discover a single pattern (Levels 1H2 and 1H3), not even the simplest patterns such as all the powers of 9 are odd or the last digit alternates between 1 and 9. This suggests that at least half of the students (or 52%) did not know how to investigate. Although the other 48% of the students did find some patterns, many of the discoveries were very trivial and this will be discussed next.

Table 1

A Rubric to Describe How Students Investigated Task 1

Level	Descriptor	No. of Students	Percentage
1H4	Students listed out the numerical values of the powers and found some patterns (quality of discovery discussed in next rubric).	14	48.3%
1H3	Students listed out the numerical values of the powers and tried to observe some patterns but no findings.	3	10.3%
1H2	Students listed out the numerical values of the powers and then did nothing.	2	6.9%
1H1	Students wrote superficial or irrelevant things that did not help in the investigation, and in particular, they did not find the numerical values of the powers but wrote things like: $9^6 = 9 \times 9 \times 9 \times 9 \times 9 \times 9$ $9^1, 9^2, 9^3, 9^4, 9^5$.	4	13.8%
1H0	Students did not do anything.	6	20.7%
	Total	**29**	**100%**

Table 2 shows the second rubric which describes *what* students investigated in the first task. There are five levels. Some of these levels are the same as the levels in the first rubric because there are overlaps: It is difficult to separate how the students investigate from what they investigate. In Level 1W0 (the letter W stands for What), the students did not do anything (the same as Level 1H0). In Level 1W1, the students did something but did not discover anything. This should be the same as the combined levels of 1H1 to 1H3 except for one student who wrote nothing constructive (i.e., Level 1H1) but conjectured some trivial patterns (i.e., Level 1W2). In Level 1W2, the students discovered or conjectured trivial patterns (which might be wrong) such as powers of 9 are divisible by 9. In Level 1W3, the students discovered or conjectured non-trivial patterns (which might be wrong) such as the last digit of the powers of 9 repeats itself after 2 times and the sum of all the digits of 9 is divisible by 9. In Level 1W4, the students discovered or conjectured complicated patterns (which might be wrong) such as the last two digits of the powers of 9 repeat themselves after 10 times. Why wrong conjectures were included in Levels 1W2 to 1W4 was because the process of formulating conjectures was also important, even if the conjectures turned out to be false. If a student discovered more than one pattern, then the level of attainment was the level of the more complicated pattern. It is important to note that *not* all the patterns described in the rubric were discovered by the students.

From the second rubric in Table 2, it was observed that 20 of the 29 students (or 69%) did not know what to investigate (Levels 1W0 to 1W2). Only nine students (or 31%) were able to find some significant patterns although some of these conjectures were false (Levels 1W3 to 1W4). In what follows, I will report some interesting pieces of information on what the non-trivial patterns or hypotheses that these nine students had found or formulated.

Table 2

A Rubric to Describe What Students Investigated for Task 1

Level	Descriptor	No. of Students	Percentage
1W4	Students discovered or conjectured more complicated patterns, which might be wrong, for example: The last two digits of the powers of 9 repeat themselves after 10 times. The last two digits of the powers of other single-digit numbers also repeats itself but after different numbers of times. When multiplying two powers of 9, the indices are added together to give the index of the resulting power of 9 (law of indices).	1	3.4%
1W3	Students discovered or conjectured non-trivial patterns, which might be wrong, for example: Powers of 9 are odd. The last digit of the powers of 9 repeats itself after 2 times. The last digit of the powers of other single-digit numbers also repeats itself but after different numbers of times. The sum of all the digits of 9 is divisible by 9. Powers of 9 are divisible by factors of 9. Odd powers of 9 contain at least one digit 9 and even powers of 9 contain no digit 9 (the latter is false since 9^{12} contains one digit 9). 9^n has n digits (which is false since 9^{22} has 21 digits).	8	27.6%
1W2	Students discovered or conjectured trivial patterns, which might be wrong, for example: Powers of 9 are divisible by 9 or are multiples of 9. When a power of 9 is divided by 9, the result is the preceding power of 9. When the index of a power of 9 is increased by 1, the new number is the same as multiplying the original power of 9 by 9.	6	20.7%
1W1	Students did some investigation but did not discover anything.	8	27.6%
1W0	Students did not do anything.	6	20.7%
	Total	**29**	**100%**

Out of these nine students (or 31%), only four of them (or 14% of the total sample) discovered that the last digit of 9 repeats itself every 2 times, and only two of them (or 7% of the total sample) examined the last digit of other powers. One of these two students, Charles, investigated the powers of 6 and the powers of 2 and concluded that the powers of different numbers have their own patterns although he did not identify the patterns. The other one, Daniel, discovered that the powers of 4 end with 4 but he made no conclusion when discussing about the powers of 5. Instead, he concluded that the pattern for the powers of 9 did not apply to the powers of 4 and the powers of 5! It seems that Daniel interpreted the suggested problem given at the end of the task statement, "Do these patterns occur for powers of other numbers?" to mean whether the exact pattern for the last digit of the powers of 9 (i.e., the repeating pattern of 1 and 9) applies to the powers of other numbers! He was unable to see beyond this specific pattern of 1 and 9 to the general pattern that the last digit of the powers of any number repeats itself. None of the students discovered the pattern in the last two digits (see Level 1W4 in Table 2).

Three of the nine students (or 10% of the total sample), Eden, Frank, and Gilbert, discovered that the sum of the digits of the powers of 9 is divisible by 9. This was rather unexpected because if the powers of 2, 4, 7, and 8 (with a period of 4 for the repeating pattern) were used in Task 1 instead of the powers of 9, then there would not have been a pattern in the sum of the digits. Eden shed some lights as to why he investigated the sum of all the digits:

> Eden: As I have read a book on Maths, I remembered that the sum of the digits in each number adds up to a multiple of 9.

Another interesting observation was made by Frank who concluded that the odd powers of 9 contain at least one digit 9 but the even powers of 9 do not contain any digit 9. Unfortunately, he stopped at the numerical value of 9^{10} because the next even power of 9, 9^{12}, contains a digit 9. It was evident that Frank did not know that the observed pattern was only a conjecture and that it could not be proven just by looking at a few empirical data.

Daniel also made the same mistake of jumping to conclusion too early. He wrote that any power of 9 had one digit more than the preceding power of 9 but he stopped at 9^{10}. If he was to observe carefully from his systematic listing of the numerical values of the powers of 9 from 9^1 to 9^{10}, he would have discovered that the first digit of the powers of 9 had decreased slowly from 9 to 3 because the next power was obtained by multiplying the preceding power by a number that was less than 10. When this list continued, the first digit would have decreased to 0 eventually, meaning that this power of 9 would have the same number of digits as the preceding power of 9, and this would eventually happen for 9^{21} and 9^{22}, both of which have 21 digits each.

One last interesting observation was by Harry. He actually discovered one of the laws of indices! The students had not studied the laws of indices and Harry's working suggests that he did not know this law beforehand because he started by trying to see if the product of two powers was another power. This was what he wrote when describing what he was doing.

> Harry: Trying to multiply the powers to see if they make out another power.

In the end, Harry discovered that $9^m \times 9^n = 9^{m+n}$. Compare this with a research study conducted by Lampert (1990). She taught a class of fifth-grade students using open tasks and mathematical discourse, which was rather similar to an investigative approach to mathematics teaching and learning. In one particular series of lessons, the students were investigating squares and they discovered some patterns in the last digit of the squares. Lampert then made use of the students' discovery to extend the problem to "What is the last digit of 5^4? 6^4? 7^4?" and later to "What is the last digit of 7^5? 7^6? 7^7? 7^8?" The students had some confusion on how to obtain 7^8 from 7^4. This would eventually lead to the discovery that the exponents should be added together when multiplying powers with the same base, which her students were almost on the verge of discovering just before the end of the series of lessons.

To summarise, only nine of the 29 students (or 31%) were able to conjecture some non-trivial patterns for Task 1. Only four out of these nine students (or 14% of the total sample) were able to discover the pattern in the last digit of the powers of 9 and only one of them (or 3% of the total sample), Charles, was able to conclude that the powers of other numbers have their own patterns in the last digit. Nevertheless, there were some interesting or unexpected findings but these were very few: only six conjectures by five students (Daniel, Eden, Frank, Gilbert, and Harry) and some of these hypotheses were even false.

For the other three investigative tasks in the test, the students fared as badly or even worse because the task statements did not provide any sample problems to investigate and so most of the students did not know how and what to investigate. It is beyond the scope of this paper to provide a detailed report of what the students did for these three tasks (these findings will be reported in another paper).

Discussion

The findings suggest that most of the high-ability students did not know how and what to investigate. The first problem is the inability of the students to pose their own problems to investigate. Many students in Singapore have not been exposed to problem posing; they are usually given problems to solve. So when the students in this study were asked to pose their own problems to solve or investigate, many of them were at a loss as to what to do.

The second problem is the failure to understand the task requirement because most students still did not know what to do even when sample problems were provided for them to investigate. Many of them did not understand what it means to find as many patterns as possible about the powers of 9. The students might be able to search for patterns to solve a specific problem but looking for *any* pattern without having a problem to solve had confounded many of them. It seems that the absence of a specific goal and the failure to understand what it means to investigate have caused them much confusion.

The third problem is jumping to conclusion too fast. For the few students who managed to make some non-trivial conjectures and even for those who made trivial conjectures by observing some patterns, many of them did not try to prove their conjectures but concluded that these were the underlying patterns based on a few empirical data, and so as it turned out, some of these hypotheses were false.

However, there were a few students who were able to make some unexpected non-trivial discoveries.

Conclusion and Implications

The current state of competency in open mathematical investigation among Singapore students is very low. If most high-ability students fared badly when given open investigative tasks, it is unlikely that low-ability and average students will know how and what to investigate but there may be some exceptions. Although further research is needed to confirm this, there are some ethical issues to consider. Many of the high-ability students gave very negative comments about the test, even to the extent that they hated it. If this test is to be administered to other classes of students, then this may cause more students to loathe mathematical investigation. This poses a serious dilemma on research methodology: How do researchers assess students' level of competency in open mathematical investigation before implementing any intervention programme? For research on problem solving, a paper-and-pencil pretest can include simpler problems at the start to give the students some confidence, and even when the students cannot solve the more difficult problems, they will at least understand what the tasks require them to do and so they will still try to solve the problems. But for research on open mathematical investigation, there is no way to make the investigative tasks any simpler. If structured tasks with part-questions are given at the start of the pretest, then these tasks are no longer open, and when the students are faced with open investigative tasks at a later part of the test, then they will still not understand the task requirements, even when sample problems are provided for the students to investigate, as the current study has shown. Thus it is necessary to rethink how to conduct research on open investigation that involves testing the subjects' initial level of proficiency.

There is also a need to expose Singapore students to this kind of open investigative tasks but anecdotal evidence (see previous section on Background) suggests that teachers are not ready to teach mathematical investigation because most of them themselves do not know how and what to investigate. Much work needs to be done to study the nature and development of thinking processes in mathematical investigation so that a suitable teaching programme can be designed to guide students to investigate effectively. Then teachers will have to be taught how to implement this programme for their students successfully.

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